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The spectral radius of tricyclic graphs with n vertices and k pendent vertices ☆

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Abstract

In this paper, we determine graphs with the largest spectral radius among all the tricyclic graphs with n vertices and k pendent vertices.

AMS classification: 05C50; 05C35

Keywords: Tricyclic graph; Spectral radius; Pendent vertex

1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let $G = (V, E)$ be a graph with n vertices and let $A(G)$ be its adjacency matrix. Since $A(G)$ is symmetric, its eigenvalues are real. Without loss of generality, we can write them as $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ and call them the eigenvalues of G. The *characteristic polynomial* of G is just $\det(\lambda I - A(G))$, denoted by $\phi(G; \lambda)$. The largest eigenvalue $\lambda_1(G)$ is called the *spectral radius* of G, denoted by $\rho(G)$. If G is connected, then $A(G)$ is irreducible and by the Perron–Frobenius theory of non-negative matrices, $\rho(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\rho(G)$. We shall refer to such an eigenvector as the *Perron vector* of G.

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The investigation on the spectral radius of graphs is an important topic in the theory of graph spectra. For results on the spectral radius of graphs, one may refer to [1–14,16,17,19–23] and the references therein.

Chang et al. [4] determined graphs with the largest spectral radius among all the bicyclic graphs on n vertices with perfect matching. Yu and Tian [22] determined the graph with the largest spectral radius among all the bicyclic graphs on n vertices with a maximum matching of cardinality m. Guo et al. [9,17] determined the graph with the largest spectral radius among all the unicyclic and bicyclic graphs with n vertices and k pendent vertices. Simić [19] determined the bicyclic graphs on prescribed number of vertices with spectral radius minimal.

Let G be a connected graph and T is a tree such that T is attached to a vertex v of G. The vertex v is called the *root* of T, or the *root-vertex* of G. Throughout this paper, we assume that T does not include the root.

A *tricyclic graph* is a connected graph in which the number of edges equals the number of vertices plus two. Denote the set of tricyclic graphs on n vertices and k pendent vertices by $\mathscr{T}_n^k$. In this paper, we study the spectral radius of tricyclic graphs on n vertices with k pendent vertices and determine the graph with the largest spectral radius in $\mathscr{T}_n^k$.

2. Preliminaries

Denote by C_n and P_n the cycle and the path, respectively, each on n vertices. Let $G-x$ or $G-xy$ denote the graph that arises from G by deleting the vertex $x \in V(G)$ or the edge $xy \in E(G)$. Similarly, $G+xy$ is a graph that arises from G by adding an edge $xy \notin E(G)$, where $x, y \in V(G)$. A *pendent vertex* of G is a vertex of degree 1. k paths $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$ are said to have *almost equal lengths* if $l_1, l_2, \ldots, l_k$ satisfy $|l_i - l_j| \leqslant 1$ for $1 \leqslant i, j \leqslant k$. For $v \in V(G)$, $d(v)$ denotes the degree of vertex v and $N(v)$ denotes the set of all neighbors of vertex v in G. We know, by [15], that a tricyclic graph G contains at least 3 cycles and at most 7 cycles, furthermore, there do not exist 5 cycles in G. Then let $\mathscr{T}_n^k = \mathscr{T}_n^{k,3} \cup \mathscr{T}_n^{k,4} \cup \mathscr{T}_n^{k,6} \cup \mathscr{T}_n^{k,7}$, where $\mathscr{T}_n^{k,i}$ denotes the set of tricyclic graphs in $\mathscr{T}_n^k$ with exact i cycles for $i = 3, 4, 6, 7$.

In this section, we list some known results which will be used in this paper.

Lemma 2.1 [16,20]. *Let G be a connected graph and $\rho(G)$ be the spectral radius of $A(G)$. Let u, v be two vertices of G and $d(v)$ be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s \in N(v)\backslash N(u)(1 \leqslant s \leqslant d(v))$ and $x = (x_1, x_2, \ldots, x_n)$ is the Perron vector of $A(G)$, where x_i corresponds to the vertex $v_i (1 \leqslant i \leqslant n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges $uv_i (1 \leqslant i \leqslant s)$. If $x_u \geqslant x_v$, then $\rho(G) < \rho(G^*)$.*

Lemma 2.2 [10]. *Let G, G', G'' be three connected graphs disjoint in pairs. Suppose that u, v are two vertices of G, u' is a vertex of G' and u'' is a vertex of G''. Let G_1 be the graph obtained from G, G', G'' by identifying, respectively, u with u' and v with u''. Let G_2 be the graph obtained from G, G', G'' by identifying vertices u, u', u''. Let G_3 be the graph obtained from G, G', G'' by identifying vertices v, u', u''. Then either $\rho(G_1) < \rho(G_2)$ or $\rho(G_1) < \rho(G_3)$.*

Let G be a connected graph, and $uv \in E(G)$. The graph $G_{u,v}$ is obtained from G by subdividing the edge uv, i.e., adding a new vertex w and edges wu, wv in $G - uv$. Hoffman and Smith define an *internal path* of G as a walk $v_0v_1 \ldots v_s (s \geqslant 1)$ such that the vertices $v_0, v_1, \ldots, v_s$ are distinct, $d(v_0) > 2$, $d(v_s) > 2$, and $d(v_i) = 2$, whenever $0 < i < s$. And s is called the length of the internal path. An internal path is closed if $v_0 = v_s$.

Let W_n be the tree on n vertices obtained from a path P_{n-4} (of length $n-5$) by attaching two new pendent edges to each end vertex of P_{n-4}, respectively. In [13], Hoffman and Smith obtained the following result:

Lemma 2.3 [13]. *Let uv be an edge of the connected graph G on n vertices.*

(i) *If uv does not belong to an internal path of G, and $G \neq C_n$, then $\rho(G_{u,v}) > \rho(G)$;*
(ii) *If uv belongs to an internal path of G, and $G \neq W_n$, then $\rho(G_{u,v}) < \rho(G)$.*

Lemma 2.4 [8,14]. *Let v be a vertex in a non-trivial connected graph G and suppose that two paths of lengths k, m, $(k \geqslant m \geqslant 1)$ are attached to G by their end vertices at v to form $G^*_{k,m}$. Then $\rho(G^*_{k,m}) > \rho(G^*_{k+1,m-1})$.*

Lemma 2.5. *Let G_1 and G_2 be two graphs.*

(i) [14] *If G_2 is a proper spanning subgraph of G_1 and G_1 is a connected graph. Then $\phi(G_2;\lambda) > \phi(G_1;\lambda)$ for $\lambda \geqslant \rho(G_1)$;*
(ii) [5,6] *If $\phi(G_2;\lambda) > \phi(G_1;\lambda)$ for $\lambda \geqslant \rho(G_2)$, then $\rho(G_1) > \rho(G_2)$;*
(iii) [13] *If G_2 is a proper subgraph of G_1 and G_1 is a connected graph, then $\rho(G_2) < \rho(G_1)$.*

Lemma 2.6 [5,18]. *Let $e = uv$ be an edge of G, and $\mathscr{C}(e)$ be the set of all cycles containing e. The characteristic polynomial of G satisfies*

$$\phi(G;\lambda) = \phi(G-e;\lambda) - \phi(G-u-v;\lambda) - 2\sum_{Z\in\mathscr{C}(e)} \phi(G\backslash V(Z);\lambda).$$

Lemma 2.7 [5]. *The characteristic polynomial of P_n satisfies the expression*

$$\phi(P_n;\lambda) = \frac{1}{\sqrt{\lambda^2-4}}(x_1^{n+1} - x_2^{n+1}),$$

where $x_1 = \frac{1}{2}(\lambda + \sqrt{\lambda^2-4})$ and $x_2 = \frac{1}{2}(\lambda - \sqrt{\lambda^2-4})$ are the roots of the equation $x^2 - \lambda x + 1 = 0$.

Lemma 2.8 [5,18]. *Let u be a vertex of G, and let $\mathscr{C}(u)$ be the set of all cycles containing u. The characteristic polynomial of G satisfies*

$$\phi(G;\lambda) = \lambda\phi(G-u;\lambda) - \sum_{v\in N(u)} \phi(G-u-v;\lambda) - 2\sum_{Z\in\mathscr{C}(u)} \phi(G\backslash V(Z);\lambda).$$

Lemma 2.9 [17]. *Let v be a vertex of G, let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of the graph G, and let $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n-1}$ be the eigenvalues of $G-v$. Then the inequalities $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n-1} \geqslant \lambda_n$ hold. If G is connected, then $\lambda_1 > \mu_1$.*

Lemma 2.10 [17]. *If the graphs G and H have exactly one eigenvalue greater than some constant a, and if $\phi(G,\rho(H)) > 0$, then $\rho(G) < \rho(H)$.*

Lemma 2.11. *If k paths on n vertices have almost equal lengths, then under the isomorphism relation the structure of this k paths is unique.*

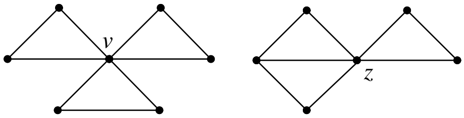

Fig. 1. Graphs G_0 and G_1.

Proof. If n is divisible by k, then our result is obviously true. Otherwise, since k paths have almost equal lengths, we can assume that k_1 paths each of which contains m vertices, the rest paths each of which contains $m-1$ vertices, where $m = \lfloor \frac{n}{k} \rfloor + 1$. Therefore,

$$n = k_1 \cdot m + (k - k_1) \cdot (m-1) = k_1 + k \cdot (m-1).$$

Note that n, k, m are fixed positive integers, therefore k_1 is determined correspondingly. This completes the proof. $\square$

Let $B_3(1)$ denote a tricyclic graph in $\mathscr{T}_n^k$ obtained from the graph G_0 in Fig. 1 by attaching k paths with almost equal lengths to vertex v.

Let $B_4(1)$ denote a tricyclic graph in $\mathscr{T}_n^k$ obtained from the graph G_1 in Fig. 1 by attaching k paths with almost equal lengths to vertex z.

Note that both $B_3(1)$ and $B_4(1)$ exist if and only if $k \geqslant 1, n \geqslant k+7$.

Lemma 2.12. *Provided that both $B_3(1)$ and $B_4(1)$ exist, the spectral radius of $B_3(1)$ is greater than that of $B_4(1)$.*

Proof. Note that, in $B_3(1)$, k paths contain exactly $n-7$ vertices, while in $B_4(1)$, k paths contain exactly $n-6$ vertices, by Lemma 2.11 we may assume that k paths in $B_3(1)$ are $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$, while those in $B_4(1)$ are $P_{l_1+1}, P_{l_2}, \ldots, P_{l_k}$. By Lemma 2.8,

$$\begin{aligned}\phi(B_3(1);\lambda) &= (\lambda^7 - 9\lambda^5 - 6\lambda^4 + 15\lambda^3 + 12\lambda^2 - 7\lambda - 6)\phi(P_{l_1};\lambda)\phi(P_{l_2};\lambda)\cdots\phi(P_{l_k};\lambda)\\ &\quad - (\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1)\sum_{i=1}^{k}\phi(P_{l_1};\lambda)\phi(P_{l_2};\lambda)\cdots\phi(P_{l_i-1};\lambda)\cdots\phi(P_{l_k};\lambda),\end{aligned} \tag{2.1}$$

$$\begin{aligned}\phi(B_4(1);\lambda) &= (\lambda^6 - 8\lambda^4 - 6\lambda^3 + 9\lambda^2 + 8\lambda)\phi(P_{l_1+1};\lambda)\phi(P_{l_2};\lambda)\cdots\phi(P_{l_k};\lambda)\\ &\quad - (\lambda^5 - 3\lambda^3 + 2\lambda)\sum_{i=1}^{k}\phi(P_{l_1+1};\lambda)\phi(P_{l_2};\lambda)\cdots\phi(P_{l_i-1};\lambda)\cdots\phi(P_{l_k};\lambda).\end{aligned} \tag{2.2}$$

Together with Eqs. (2.1) and (2.2), we have

$$\phi(B_4(1);\lambda) - \phi(B_3(1);\lambda) = f(\lambda) + g(\lambda),$$

where

$$\begin{aligned}f(\lambda) &= (\lambda^6 - 8\lambda^4 - 6\lambda^3 + 9\lambda^2 + 8\lambda)\phi(P_{l_1+1};\lambda)\phi(P_{l_2};\lambda)\cdots\phi(P_{l_k};\lambda)\\ &\quad - (\lambda^5 - 3\lambda^3 + 2\lambda)\cdot\phi(P_{l_1};\lambda)\phi(P_{l_2};\lambda)\cdots\phi(P_{l_k};\lambda)\\ &\quad - (\lambda^7 - 9\lambda^5 - 6\lambda^4 + 15\lambda^3 + 12\lambda^2 - 7\lambda - 6)\phi(P_{l_1};\lambda)\cdot\phi(P_{l_2};\lambda)\cdots\phi(P_{l_k};\lambda)\end{aligned}$$

$$+ (\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1)\phi(P_{l_1-1};\lambda)\phi(P_{l_2};\lambda)\cdots\phi(P_{l_k};\lambda),$$

$$g(\lambda) = (\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1)\sum_{i=2}^{k}\phi(P_{l_1};\lambda)\cdots\phi(P_{l_i-1};\lambda)\cdots\phi(P_{l_k};\lambda)$$

$$- (\lambda^5 - 3\lambda^3 + 2\lambda)\sum_{i=2}^{k}\phi(P_{l_1+1};\lambda)\phi(P_{l_2};\lambda)\cdots\phi(P_{l_i-1};\lambda)\cdots\phi(P_{l_k};\lambda).$$

Furthermore, repeatedly using Lemma 2.8 we can simplify both $f(\lambda)$ and $g(\lambda)$ as following:

$$f(\lambda) = (2\lambda^4 + 2\lambda^3 - \lambda^2 - 2\lambda - 1)\phi(P_{l_1-1};\lambda)\phi(P_{l_2};\lambda)\cdots\phi(P_{l_k};\lambda)$$
$$+ (3\lambda^3 + 4\lambda^2 - 5\lambda - 6)\phi(P_{l_1-2};\lambda)\phi(P_{l_2};\lambda)\cdots\phi(P_{l_k};\lambda),$$

$$g(\lambda) = \sum_{i=2}^{k}\{(\lambda^2 - 1)\phi(P_{l_1};\lambda)\cdots\phi(P_{l_i-1};\lambda)\cdots\phi(P_{l_k};\lambda)$$
$$+ (\lambda^5 - 3\lambda^3 + 2\lambda)\phi(P_{l_1-1};\lambda)\phi(P_{l_2};\lambda)\cdots\phi(P_{l_i-1};\lambda)\cdots\phi(P_{l_k};\lambda)\}.$$

Note that, when $\lambda \geq \rho(B_4(1))$, by (iii) in Lemma 2.5 we have

$$\phi(P_{l_1-2};\lambda)\phi(P_{l_2};\lambda)\cdots\phi(P_{l_k};\lambda) > 0,$$

and

$$\phi(P_{l_1-1};\lambda)\phi(P_{l_2};\lambda)\cdots\phi(P_{l_i-1};\lambda)\cdots\phi(P_{l_k};\lambda) > 0$$

for all $i = 2, 3, \ldots, k$.

On the other hand, note that K_3 is a proper subgraph of $B_4(1)$, by (iii) in Lemma 2.5, we have $\rho(B_4(1)) > \rho(K_3) = 2$. Hence, when $\lambda \geqslant \rho(B_4(1)) > 2$, it is easy to see

$$2\lambda^4 + 2\lambda^3 - \lambda^2 - 2\lambda - 1 > 0, \quad 3\lambda^3 + 4\lambda^2 - 5\lambda - 6 > 0,$$
$$\lambda^2 - 1 > 0, \quad \lambda^5 - 3\lambda^3 + 2\lambda > 0.$$

Therefore, $f(\lambda) + g(\lambda) > 0$, i.e., $\phi(B_4(1);\lambda) > \phi(B_3(1);\lambda)$ for $\lambda \geqslant \rho(B_4(1))$. By Lemma 2.5, $\rho(B_3(1)) > \rho(B_4(1))$. This completes the proof. □

Let $B_7(1)$ be a tricyclic graph in $\mathscr{T}_n^k$ created from K_4 by attaching k paths with almost equal lengths to a vertex, say v, of K_4. Applying Lemma 2.8 to vertex v of $B_7(1)$, we have

$$\phi(B_7(1);\lambda) = (\lambda^4 - 6\lambda^2 - 8\lambda - 3)\phi(P_{l_1};\lambda)\cdots\phi(P_{l_k};\lambda)$$
$$- (\lambda^3 - 3\lambda - 2)\sum_{i=1}^{k}\phi(P_{l_1};\lambda)\cdots\phi(P_{l_i-1};\lambda)\cdots\phi(P_{l_k};\lambda).$$

Note that both $B_3(1)$ and $B_7(1)$ exist if and only if $k \geqslant 1, n \geqslant k + 7$.

Lemma 2.13. *Provided that both $B_3(1)$ and $B_7(1)$ exist, the spectral radius of $B_3(1)$ is greater than that of $B_7(1)$.*

Proof. Denote by l, ($l \geqslant 2$) the maximal number of vertices of a path attached to the vertex v of $B_7(1)$. Suppose that the number of such paths is t.

Case 1. $t \geqslant 3$.

Let B_3 be the graph analogous to $B_3(1)$ in which all paths attached to vertex v have $l-1$ vertices. Let B_7 be the graph analogous to $B_7(1)$ in which all paths attached to vertex v have l vertices. Evidently, B_3 is an induced subgraph of $B_3(1)$ whereas $B_7(1)$ is an induced subgraph of B_7. Therefore, by Lemma 2.5,

$$\rho(B_3) \leqslant \rho(B_3(1))$$

with equality if and only if $n = (l-1)k + 7$. Also,

$$\rho(B_7) \geqslant \rho(B_7(1))$$

with equality if and only if $n = lk + 4$.

Thus for the proof of Lemma 2.13 it is sufficient to show that $\rho(B_7) < \rho(B_3)$. We do this in the following.

Because of Lemma 2.9, the graphs B_3 and B_7 have exactly one eigenvalue greater than 3. (This is because all components of the subgraphs $B_3 - v$ and $B_7 - v$ are paths and K_3, and the spectral radii of paths are less than 2, while the spectral radius of K_3 is 2. Therefore $\lambda_2(B_3) < 3$ and $\lambda_2(B_7) < 3$. By direct calculation we check that in the case $n = 10, k = 3$, the greatest eigenvalues of B_3 and B_7 are greater than 3. Therefore the greatest eigenvalues of B_3 and B_7 are greater than 3 for all values of n and k.) Consequently, Lemma 2.10 is applicable to B_3 and B_7 and it is sufficient to show that $\phi(B_7; \rho(B_3)) > 0$.

By applying Lemma 2.8 to the vertex v of B_3 and B_7, respectively, we obtain

$$\begin{aligned}\phi(B_3;\lambda) &= (\lambda^7 - 9\lambda^5 - 6\lambda^4 + 15\lambda^3 + 12\lambda^2 - 7\lambda - 6)\phi(P_{l-1};\lambda)^k \\ &\quad - k(\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1)\phi(P_{l-1};\lambda)^{k-1}\phi(P_{l-2};\lambda), \\ \phi(B_7;\lambda) &= (\lambda^4 - 6\lambda^2 - 8\lambda - 3)\phi(P_l;\lambda)^k - k(\lambda^3 - 3\lambda - 2)\phi(P_l;\lambda)^{k-1}\phi(P_{l-1};\lambda).\end{aligned}$$

Denote the greatest eigenvalue of B_3 by r. For $n = 10$ and $k = 3$ the greatest eigenvalue of $B_3(1)$ is 3.3926. Therefore, for any n and k, $r = \rho(B_3) \geqslant 3.3923$.

From the above expression for $\phi(B_3;\lambda)$ it is seen that r satisfies the equation

$$\begin{aligned}&(r^7 - 9r^5 - 6r^4 + 15r^3 + 12r^2 - 7r - 6)\phi(P_{l-1};r) \\ &\quad - k(r^6 - 3r^4 + 3r^2 - 1)\phi(P_{l-2};r) = 0\end{aligned}$$

from which

$$k = \frac{(r^7 - 9r^5 - 6r^4 + 15r^3 + 12r^2 - 7r - 6)\phi(P_{l-1};r)}{(r^6 - 3r^4 + 3r^2 - 1)\phi(P_{l-2};r)}.$$

Now, the inequality $\phi(B_7;r) > 0$ holds if and only if

$$r\phi(P_l;r)^{k-1}[(r^4 - 6r^2 - 8r - 3)\phi(P_l;r) - k(r^3 - 3r - 2)\phi(P_{l-1};r)] > 0$$

if and only if

$$(r^4 - 6r^2 - 8r - 3)\phi(P_l;r) > k(r^3 - 3r - 2)\phi(P_{l-1};r)$$

if and only if

$$\begin{aligned}&(r^4 - 6r^2 - 8r - 3)\phi(P_l;r) \\ &\quad > \frac{(r^7 - 9r^5 - 6r^4 + 15r^3 + 12r^2 - 7r - 6)\phi(P_{l-1};r)}{(r^6 - 3r^4 + 3r^2 - 1)\phi(P_{l-2};r)}(r^3 - 3r - 2)\phi(P_{l-1};r)\end{aligned}$$

if and only if

$$(r^{10}-9r^8-8r^7+18r^6+24r^5-10r^4-24r^3-3r^2+8r+3)\phi(P_l;r)\phi(P_{l-2};r)$$
$$> (r^{10}-12r^8-8r^7+42r^6+48r^5-40r^4-72r^3-3r^2+32r+12)\phi(P_{l-1};r)^2.$$

From Lemma 2.7 we get

$$\phi(P_n;r)=\frac{1}{\sqrt{r^2-4}}(r_1^{n+1}-r_2^{n+1}),$$

where $r_1=\frac{1}{2}(r+\sqrt{r^2-4})$ and $r_2=\frac{1}{2}(r-\sqrt{r^2-4})$ are the roots of the equation $x^2-rx+1=0$. From the Vieta formulas, $r_1+r_2=r$; $r_1r_2=1$ and therefore

$$r_1^2+r_2^2=(r_1+r_2)^2-2r_1r_2=r^2-2,$$
$$r_1^4+r_2^4=(r_1^2+r_2^2)^2-2r_1^2r_2^2=(r^2-2)^2-2.$$

In view of the above, $\phi(B_7;r)>0$ holds if and only if

$$\begin{aligned}&\frac{1}{r^2-4}(r^{10}-9r^8-8r^7+18r^6+24r^5-10r^4\\&\quad-24r^3-3r^2+8r+3)(r_1^{l+1}-r_2^{l+1})(r_1^{l-1}-r_2^{l-1})\\&\quad>\frac{1}{r^2-4}(r^{10}-12r^8-8r^7+42r^6+48r^5-40r^4\\&\quad-72r^3-3r^2+32r+12)(r_1^l-r_2^l)^2\end{aligned}$$

if and only if

$$\begin{aligned}&(r_1^{2l}+r_2^{2l})(3r^8-24r^6-24r^5+30r^4+48r^3-24r-9)\\&\quad>r^{12}-13r^{10}-8r^9+60r^8+56r^7-130r^6-168r^5\\&\quad+97r^4+200r^3+15r^2-80r-30.\end{aligned}\tag{2.3}$$

We now demonstrate that for $l\geqslant 2$ the series $a_l=r_1^{2l}+r_2^{2l}$ strictly increases. Because $r_1^{2l}+r_2^{2l}=\frac{r_1^{4l}+1}{r_1^{2l}}$, we get that

$$\frac{a_{l+1}}{a_l}=\frac{r_1^{4l+4}+1}{r_1^{4l+2}+r_1^2}$$

will be greater than unity (in which case a_l increases) if and only if

$$r_1^{4l+4}+1>r_1^{4l+2}+r_1^2$$

i.e., if

$$(r_1^{4l+2}-1)(r_1^2-1)>0$$

which is evidently obeyed since $r_1>1$.

We have previously shown that $\phi(B_7;r)>0$ holds if and only if Inequality (2.3) holds. Now, if Inequality (2.3) is satisfied for $l=2$ it will be satisfied for all $l\geqslant 2$.

For $l=2$ we get

$$(r_1^4+r_2^4)(3r^8-24r^6-24r^5+30r^4+48r^3-24r-9)$$

$$\begin{aligned}&> r^{12} - 13r^{10} - 8r^9 + 60r^8 + 56r^7 - 130r^6 - 168r^5 \\ &\quad + 97r^4 + 200r^3 + 15r^2 - 80r - 30\end{aligned}$$

if and only if

$$\begin{aligned}&2r^{12} - 23r^{10} - 16r^9 + 72r^8 + 88r^7 - 38r^6 - 96r^5 \\ &\quad - 46r^4 - 8r^3 + 21r^2 + 32r + 12 > 0.\end{aligned}$$

Let

$$\begin{aligned}f(r) = {}& 2r^{12} - 23r^{10} - 16r^9 + 72r^8 + 88r^7 - 38r^6 \\ & - 96r^5 - 46r^4 - 8r^3 + 21r^2 + 32r + 12.\end{aligned}$$

Note that $r > 3$, we have $2r^{12} > \frac{5}{3}r^{12} + r^{11}$. Therefore,

$$\begin{aligned}f(r) &> \frac{5}{3}r^{12} + r^{11} - 23r^{10} - 16r^9 + 72r^8 + 88r^7 - 38r^6 \\ &\quad - 96r^5 - 46r^4 - 8r^3 + 21r^2 + 32r + 12 \\ &= \left(\frac{5}{3}r^{12} - 23r^{10} + 67r^8\right) + (r^{11} - 16r^9 + 64r^7) + (5r^8 - 38r^6 - 46r^4) \\ &\quad + (24r^7 - 96r^5 - 8r^3) + (21r^2 + 32r + 12).\end{aligned}$$

It is easy to show when $r > 3.1019$, that $\frac{5}{3}r^{12} - 23r^{10} + 67r^8 > 0$, $r^{11} - 16r^9 + 64r^7 > 0$, $5r^8 - 38r^6 - 46r^4 > 0$, $24r^7 - 96r^5 - 8r^3 > 0$, $21r^2 + 32r + 12 > 0$. Namely, when $r > 3.1019$, $f(r) > 0$. Note that $r = \rho(B_3) \geqslant 3.3923$, and so, we have demonstrated that $\phi(B_7; \rho(B_3)) > 0$, which, by Lemma 2.10, implies

$$\rho(B_3) > \rho(B_7).$$

Therefore,

$$\rho(B_3(1)) > \rho(B_7(1)).$$

Case 2. $1 \leqslant t \leqslant 2$, $k \geqslant 2$.

In this case, it is straightforward to check that the maximal number of vertices of a path attached to the vertex v of $B_7(1)$ is l, while the minimal number of vertices of a path attached to the vertex v of $B_3(1)$ is $l - 2$, where $l \geqslant 3$. Thus, let B_3' be the graph analogous to $B_3(1)$ in which all paths attached to vertex v have $l - 2$ vertices, whereas let B_7 be the graph analogous to $B_7(1)$ in which all paths attached to vertex v have l vertices.

Evidently, B_3' is an induced subgraph of $B_3(1)$ whereas $B_7(1)$ is an induced subgraph of B_7. Therefore, by Lemma 2.5,

$$\rho(B_3') \leqslant \rho(B_3(1))$$

with equality if and only if $n = (l - 2)k + 7$. Also,

$$\rho(B_7) \geqslant \rho(B_7(1))$$

with equality if and only if $n = lk + 4$.

Thus for the proof of Lemma 2.13 it is sufficient to show that $\rho(B_7) < \rho(B_3')$. We do this in the following.

Because of Lemma 2.9, the graphs B_3' and B_7 have exactly one eigenvalue greater than 3. (This is because all components of the subgraphs $B_3' - v$ and $B_7 - v$ are paths and K_3, and the spectral radii of paths are less than 2, while the spectral radius of K_3 is 2. Therefore $\lambda_2(B_3') < 3$ and $\lambda_2(B_7) < 3$. By direct calculation we check that in the case $n = 9$, $k = 2$, the greatest eigenvalues of B_3' and B_7 are greater than 3. Therefore the greatest eigenvalues of B_3' and B_7 are greater than 3 for all values of n and k.) Consequently, Lemma 2.10 is applicable to B_3' and B_7 and it is sufficient to show that $\phi(B_7; \rho(B_3')) > 0$.

By applying Lemma 2.8 to the vertex v of B_3' and B_7, respectively, we obtain

$$\begin{aligned}\phi(B_3'; \lambda) &= (\lambda^7 - 9\lambda^5 - 6\lambda^4 + 15\lambda^3 + 12\lambda^2 - 7\lambda - 6)\phi(P_{l-2}; \lambda)^k \\ &\quad - k(\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1)\phi(P_{l-2}; \lambda)^{k-1}\phi(P_{l-3}; \lambda),\\ \phi(B_7; \lambda) &= (\lambda^4 - 6\lambda^2 - 8\lambda - 3)\phi(P_l; \lambda)^k - k(\lambda^3 - 3\lambda - 2)\phi(P_l; \lambda)^{k-1}\phi(P_{l-1}; \lambda).\end{aligned}$$

Denote the greatest eigenvalue of B_3' by r. For $n = 9$ and $k = 2$ the greatest eigenvalue of B_3' is 3.2635. Therefore, for any n and k, $r = \rho(B_3') \geqslant 3.2635$.

Similarly to Case 1, $\phi(B_7; r) > 0$ holds if and only if

$$\begin{aligned}&2r^{13} - 26r^{11} - 16r^{10} + 108r^9 + 112r^8 - 164r^7 - 240r^6\\ &\quad + 74r^5 + 208r^4 + 30r^3 - 64r^2 - 24r > 0.\end{aligned}$$

Let

$$\begin{aligned}f(r) &= 2r^{12} - 26r^{10} - 16r^9 + 108r^8 + 112r^7 - 164r^6 - 240r^5\\ &\quad + 74r^4 + 208r^3 + 30r^2 - 64r - 24\\ &= (2r^{12} - 26r^{10} - 16r^9 + 92r^8 + 88r^7) + (16r^8 - 164r^6 + 74r^4)\\ &\quad + (24r^7 - 240r^5 + 144r^3) + (64r^3 - 64r) + (30r^2 - 24).\end{aligned}$$

It is easy to show when $r > 3.126$, that $16r^8 - 164r^6 + 74r^4 > 0$, $24r^7 - 240r^5 + 144r^3 > 0$, $64r^3 - 64r > 0$, $30r^2 - 24 > 0$.

Let

$$g(r) = 2r^{12} - 26r^{10} - 16r^9 + 92r^8 + 88r^7.$$

When $r > 3$, $g'(r) = 24r^{11} - 260r^9 - 144r^8 + 736r^7 + 616r^6 > 0$, this shows that $g(r)$ strictly increases when $r > 3$. Note that $g(3) > 0$, and so $g(r) > 0$ when $r > 3$. Namely, when $r > 3.1019$, $f(r) > 0$. Recall that $r = \rho(B_3') \geqslant 3.2635$, therefore, we have demonstrated that $\phi(B_7; \rho(B_3')) > 0$, which, by Lemma 2.10, implies

$$\rho(B_3') > \rho(B_7).$$

Therefore,

$$\rho(B_3(1)) > \rho(B_7(1)).$$

Case 3. $t = 1$, $k = 1$. In this case, we have

$$\begin{aligned}\phi(B_3(1); \lambda) &= (\lambda^7 - 9\lambda^5 - 6\lambda^4 + 15\lambda^3 + 12\lambda^2 - 7\lambda - 6)\phi(P_{l-3}; \lambda)\\ &\quad - (\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1)\phi(P_{l-4}; \lambda),\\ \phi(B_7(1); \lambda) &= (\lambda^4 - 6\lambda^2 - 8\lambda - 3)\phi(P_l; \lambda) - (\lambda^3 - 3\lambda - 2)\phi(P_{l-1}; \lambda).\end{aligned}$$

Because of Lemma 2.9, the graphs $B_3(1)$ and $B_7(1)$ have exactly one eigenvalue greater than 3. (This is because all components of the subgraphs $B_3(1) - v$ and $B_7(1) - v$ are paths and K_3, and the spectral radii of paths are less than 2, while the spectral radius of K_3 is 2. Therefore $\lambda_2(B_3(1)) < 3$ and $\lambda_2(B_7(1)) < 3$. By direct calculation we check that in the case $n = 8, k = 1$, the greatest eigenvalues of $B_3(1)$ and $B_7(1)$ are greater than 3. Therefore the greatest eigenvalues of $B_3(1)$ and $B_7(1)$ are greater than 3 for all values of n and k.) Consequently, Lemma 2.10 is applicable to $B_3(1)$ and $B_7(1)$ and it is sufficient to show that $\phi(B_7(1); \rho(B_3(1))) > 0$.

Denote the greatest eigenvalue of $B_3(1)$ by r. For $n = 8$ and $k = 1$ the greatest eigenvalue of $B_3(1)$ is 3.1326. Therefore, for any n and k,

$$r = \rho(B_3(1)) \geqslant 3.1326.$$

From the above expression for $\phi((B_3(1)); \lambda)$ it is seen that r satisfies the equation:

$$(r^7 - 9r^5 - 6r^4 + 15r^3 + 12r^2 - 7r - 6)\phi(P_{l-3}; r) \\ - (r^6 - 3r^4 + 3r^2 - 1)\phi(P_{l-4}; r) = 0$$

from which

$$\frac{\phi(P_{l-4}; r)}{\phi(P_{l-3}; r)} = \frac{r^7 - 9r^5 - 6r^4 + 15r^3 + 12r^2 - 7r - 6}{r^6 - 3r^4 + 3r^2 - 1}.$$

Now, the inequality $\phi(B_7(1); r) > 0$ hold if and only if

$$(r^4 - 6r^2 - 8r - 3)\phi(P_l; r) - (r^3 - 3r - 2)\phi(P_{l-1}; r) > 0.$$

Note that

$$\phi(P_l; r) = (r^3 - 2r)\phi(P_{l-3}; r) - (r^2 - 1)\phi(P_{l-4}; r), \\ \phi(P_{l-1}; r) = (r^2 - 1)\phi(P_{l-3}; r) - r\phi(P_{l-4}; r),$$

so we have

$$\phi(B_7(1); r) = (r^7 - 9r^5 - 8r^4 + 13r^3 + 18r^2 + 3r - 2)\phi(P_{l-3}; r) \\ - (r^6 - 8r^4 - 8r^3 + 6r^2 + 10r + 3)\phi(P_{l-4}; r).$$

That is to say, $\phi(B_7(1); r) > 0$ if and only if

$$(r^7 - 9r^5 - 8r^4 + 13r^3 + 18r^2 + 3r - 2)(r^6 - 3r^4 + 3r^2 - 1) \\ > (r^7 - 9r^5 - 6r^4 + 15r^3 + 12r^2 - 7r - 6)(r^6 - 8r^4 - 8r^3 + 6r^2 + 10r + 3)$$

if and only if

$$5r^{11} + 6r^{10} - 50r^9 - 100r^8 + 66r^7 + 268r^6 + 76r^5 \\ - 240r^4 - 175r^3 + 46r^2 + 78r + 20 > 0.$$

Let

$$f(r) = 5r^{11} + 6r^{10} - 50r^9 - 100r^8 + 66r^7 + 268r^6 \\ + 76r^5 - 240r^4 - 175r^3 + 46r^2 + 78r + 20.$$

Similarly to the proof of Cases 1 and 2, we can also show that $f(r) > 0$, when $r = \rho(B_3(1))$, which, by Lemma 2.10, implies $\rho(B_3(1)) > \rho(B_7(1))$.

Combining Cases 1–3, we completes the proof. $\square$

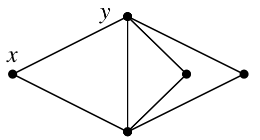

Fig. 2. Graph G_0.

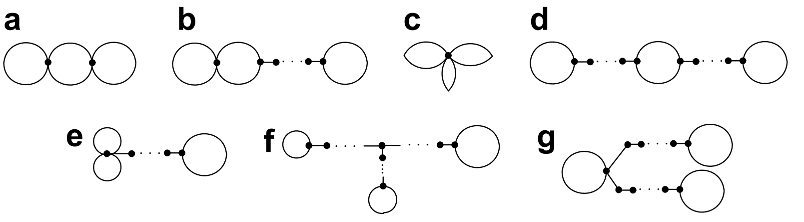

Fig. 3. Seven possible cases for the arrangement of three cycles in G.

Let $B_6(1)$ (respectively, $B_6(2)$) denote a tricyclic graph in $\mathscr{T}_n^k$ obtained from the graph G_0 in Fig. 2 by attaching k paths with almost equal lengths to vertex y (respectively, x).

With the same method used in Lemma 2.13, we can prove the following lemma, we will not repeat the procedure here. Note that both $B_3(1)$ and $B_6(1)$ exist if and only if $k \geqslant 1, n \geqslant k+7$.

Lemma 2.14. *Provided that both $B_3(1)$ and $B_6(1)$ exist, the spectral radius of $B_3(1)$ is greater than that of $B_6(1)$.*

3. Main results

Theorem 3.1. *Let G be a graph in $\mathscr{T}_n^{k,3}$, $k \geqslant 1$. Then $\rho(G) \leqslant \rho(B_3(1))$, and the equality holds if and only if $G \cong B_3(1)$.*

Proof. The arrangement of three cycles, say C_p, C_q, C_h, in G has seven possible cases; see Fig. 3. Choose $G \in \mathscr{T}_n^{k,3}$ such that the spectral radius of G is as large as possible. Denote the vertex set of G by $\{v_1, v_2, \ldots, v_n\}$ and the Perron vector of G by $x = (x_1, x_2, \ldots, x_n)$, where x_i corresponds to the vertex $v_i (1 \leqslant i \leqslant n)$. We first prove some facts.

Fact 1. The arrangement of three cycles in G is (c) in Fig. 3.

Proof of Fact 1. Assume that the arrangement of the three cycles contained in G is just (b) in Fig. 3. Then denote the path connecting two cycles, say C_p, C_q, by $v_1 v_2 \ldots v_l$ with $l > 1$. Suppose that v_1 is on C_p, while v_l is on C_q. Without loss of generality, we may assume that $x_1 \geqslant x_l$. Denote by v_{l+1} a neighbor of v_l which belongs to C_q. Let

$$G^* = G - \{v_l v_{l+1}\} + \{v_1 v_{l+1}\}.$$

Then $G^* \in \mathscr{T}_n^{k,3}$. By Lemma 2.1, we have $\rho(G^*) > \rho(G)$, a contradiction. Hence $l = 1$. Using the same method, we can also show that G cannot contains three cycles whose arrangement is as (d), (e), (f) or (g) in Fig. 3. Therefore the arrangement of three cycles in G is either (a) or (c) in

Fig. 3. By Lemma 2.2, we know that G cannot contains three cycles whose arrangement is that of (a). This completes the proof of Fact 1.

Furthermore, by Fact 1, we can prove by Lemma 2.2 that G has exactly one tree, say T, attached and the root, say v, of the tree is the common vertex of the three cycles.

Fact 2. Each vertex u of T has degree $d(u) \leqslant 2$.

Proof of Fact 2. On the contrary, if there exists one vertex v_i of T such that $d(v_i) > 2$. Denote $N(v_i) = \{z_1, z_2, \ldots, z_t\}$ and $N(v) = \{w_1, w_2, \ldots, w_s\}$. Assume that z_1, w_3 belong to the path joining v and v_i, and that w_1 (respectively, w_2) belongs to some cycle in G. If $x_v \geqslant x_i$, let

$$G_3^* = G - \{v_i z_3, \ldots, v_i z_t\} + \{v z_3, \ldots, v z_t\}.$$

If $x_v < x_i$, let

$$G_4^* = G - \{v w_1, v w_4, \ldots, v w_s\} + \{v_i w_1, v_i w_4, \ldots, v_i w_s\}.$$

Then $G_3^*, G_4^* \in \mathscr{T}_n^{k,3}$. By Lemma 2.1, we have $\rho(G_3^*) > \rho(G)$ and $\rho(G_4^*) > \rho(G)$, a contradiction. Hence G is a graph with k paths attached to v.

Fact 3. k paths attached to v have almost equal lengths.

Proof of Fact 3. Denote the k paths attached to v by $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$, then we will show that $|l_i - l_j| \leqslant 1$ for $1 \leqslant i, j \leqslant k$. If there exist two paths, say P_{l_1} and P_{l_2}, such that $l_1 - l_2 \geqslant 2$. Denote $P_{l_1} = v u_1 u_2 \ldots u_{l_1}$ and $P_{l_2} = v w_1 w_2 \ldots w_{l_2}$. Let

$$G^* = G - \{u_{l_1 - 1} u_{l_1}\} + \{w_{l_2} u_{l_1}\}.$$

Then $G^* \in \mathscr{T}_n^{k,3}$. By Lemma 2.4, we have $\rho(G^*) > \rho(G)$, a contradiction.

Fact 4. All the cycles C_p, C_q and C_h in G have length 3.

Proof of Fact 4. Assume that $p \geqslant 4$. Let $C_p = v v_1 v_2 \ldots v_{p-1} v$ and let $P_m = v u_1 u_2 \ldots u_m$ be a path attached to the graph G, where $m \geqslant 1$. Obviously, $G \neq C_n$, $G \neq W_n$, $v u_1 u_2 \ldots u_m$ is not an internal path.

Let

$$G^* = G - \{v v_1, v_1 v_2\} + \{v v_2, u_m v_1\}.$$

Then $G^* \in \mathscr{T}_n^{k,3}$. By Lemma 2.3, we have $\rho(G^*) > \rho(G)$, a contradiction. Hence $p = 3$. Similarly, we can verify that $q = 3$ and $h = 3$.

Combining Facts 1–4, we have $G = B_3(1)$. This completes the proof. $\square$

Theorem 3.2. *Let G be a graph in $\mathscr{T}_n^{k,4}$, $k \geqslant 1$. Then $\rho(G) \leqslant \rho(B_4(1))$, and the equality holds if and only if $G \cong B_4(1)$.*

Proof. Let $P_{l+1}, P_{p+1}, P_{q+1}$ be three vertex-disjoint paths, where $l, p, q \geqslant 1$ and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, respectively, the resulting graph (e.g., (i) in Fig. 4), denoted by $P(l, p, q)$, is called a θ-graph. Furthermore, let C_h be a cycle. Connect C_h and $P(l, p, q)$ by a path P_s and denote the resulting graph by G', where $s \geqslant 1$ and G' has four types, see (ii)–(v) in Fig. 4. So, $\mathscr{T}_n^{k,4}$ are those graphs each of which is obtained by attaching some trees to G'.

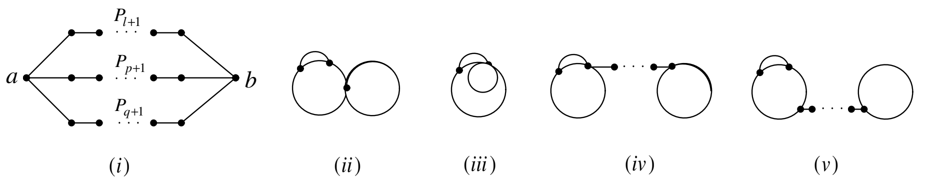

Fig. 4. (i) is the graph $P(l, p, q)$ and (ii), (iii), (iv), (v) are the four types of G'.

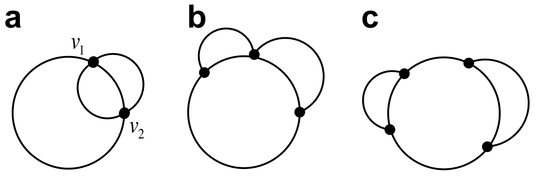

Fig. 5. Three possible cases for the arrangement of six cycles in G.

Choose $G \in \mathscr{T}_n^{k,4}$ such that the spectral radius of G is as large as possible. Denote the vertex set of G by $\{v_1, v_2, \ldots, v_n\}$ and the Perron vector of G by $x = (x_1, x_2, \ldots, x_n)$, where x_i corresponds to the vertex $v_i (1 \leqslant i \leqslant n)$.

Similarly to the proof of Theorem 3.1 we can verify that G is a G'-graph with k paths of almost equal lengths attached to one vertex denoted by v, where G' is either Type (ii) or Type (iii) and the cycle C_h is of length 3. We will use $P_m = vu_1u_2 \cdots u_m$ to denote one of the k paths attached to vertex v, where $m \geqslant 1$.

We can prove by Lemma 2.1 that G' must be of Type (iii). For convenience, we assume that a is just the common vertex of $P(l, p, q)$ and C_h. Then by Lemma 2.2 we obtain that $v = a$.

By the definition of graph $P(l, p, q)$, we have $l, p, q \geqslant 1$ and at most one of them is 1. We claim that one of p, q, l is 1 and the other two are 2. Assume, on the contrary, that $l \geqslant 3$. Put $P_{l+1} = vv_2 \cdots v_{l+1}$. Obviously, $G \neq C_n$, $G \neq W_n$, $vv_2 \cdots v_{l+1}$ is an internal path, and $vu_1u_2 \cdots u_m$ is not an internal path. Let

$$G^* = G - \{vv_2, v_2v_3\} + \{vv_3, u_mv_2\}.$$

Then $G^* \in \mathscr{T}_n^{k,4}$. By Lemma 2.3, we have $\rho(G^*) > \rho(G)$, a contradiction. Hence $l \leqslant 2$. Similarly, we can verify that $p, q \leqslant 2$ and that one and only one of l, p, q is 1. Thus $G = B_4(1)$. $\square$

Theorem 3.3. *Let G be a graph in $\mathscr{T}_n^{k,6}$, $k \geqslant 1$. Then $\rho(G) \leqslant \rho(B_6(1))$, and the equality holds if and only if $G \cong B_6(1)$.*

Proof. Since G contains exactly six cycles, then it is straightforward to check that all of the six cycles either have exactly two vertices in common, or have exactly one vertex in common, or have no vertex in common; see Fig. 5.

Choose $G \in \mathscr{T}_n^{k,6}$ such that the spectral radius of G is as large as possible. Denote the vertex set of G by $\{v_1, v_2, \ldots, v_n\}$ and the Perron vector of G by $x = (x_1, x_2, \ldots, x_n)$, where x_i corresponds to the vertex $v_i (1 \leqslant i \leqslant n)$.

Similarly to the proof of Theorem 3.1 we can verify that there exactly exist k paths of almost equal lengths attached to one vertex, say v, on a cycle of G.

We can prove by Lemma 2.1 that the six cycles contained in G must be (a) in Fig. 5. Then label the common vertices of the six cycles as v_1, v_2, which are depicted in Fig. 5 (a).

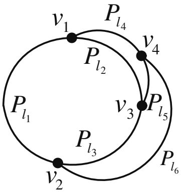

Fig. 6. One possible case for the arrangement of seven cycles in G.

Assume that $P_{l+1}, P_{p+1}, P_{q+1}, P_{h+1}$ are the four paths connecting v_1 and v_2; see (a) in Fig. 5. Similarly to the proof of Theorem , we obtain that one of l, p, q, h is 1 and the other three are 2. When $v = v_1$ or v_2, then $G = B_6(1)$. When $v \notin \{v_1, v_2\}$, then $G = B_6(2)$.

Now we show that $\rho(B_6(1)) > \rho(B_6(2))$ in the following.

In fact, denote one of the vertices of degree 4 in $B_6(2)$ by u, let $\{u_1, u_2\} \in N(u)$, where $\{u_1, u_2\} \notin N(x)$ and $x \notin \{u_1, u_2\}$. Denote $N(x) = \{v_1, v_2, \ldots, v_k, u, w\}$, where $w \in N(x) \cap N(u)$. If $x_u > x_x$, let

$$G_1^* = B_6(2) - \{xv_1, xv_2, \ldots, xv_k\} + \{uv_1, uv_2, \ldots, uv_k\}.$$

If $x_u < x_x$, let

$$G_2^* = B_6(2) - \{uu_1, uu_2\} + \{xu_1, xu_2\}.$$

Then $G_1^* = G_2^* = B_6(1)$. By Lemma 2.1, we have $\rho(B_6(1)) > \rho(B_6(2))$. This completes the proof. □

Theorem 3.4. *Let G be a graph in $\mathscr{T}_n^{k,7}$, $k \geqslant 1$. Then $\rho(G) \leqslant \rho(B_7(1))$, and the equality holds if and only if $G \cong B_7(1)$.*

Proof. The arrangement of seven cycles contained in G has only one case; see Fig. 6. Choose $G \in \mathscr{T}_n^{k,7}$ such that the spectral radius of G is as large as possible. Denote the vertex set of G by $\{v_1, v_2, \cdots, v_n\}$ and the Perron vector of G by $x = (x_1, x_2, \cdots, x_n)$, where x_i corresponds to the vertex $v_i (1 \leqslant i \leqslant n)$.

Similarly to the proof of Theorem 3.1 we can verify that there exactly exist k paths of almost equal lengths attached to one vertex, say v, on a cycle of G. For convenience, v_1, v_2, v_3, v_4 and $P_{l_1}, \ldots, P_{l_6}$ are just as shown in Fig. 6. We will use $P_m = vw_1 \cdots w_m$ to denote one of the k paths attached to vertex v, where $m \geqslant 1$. We can prove by Lemma 2.1 that v is in $\{v_1, v_2, v_3, v_4\}$.

Now we want to show that $l_1 = l_2 = \cdots = l_6 = 2$. Assume, on the contrary, that $l_1 \geqslant 3$, then let $P_{l_1} = v_1u_1u_2 \ldots u_s$ and $u_s = v_2$ and $s \geqslant 2$. Obviously, $G \neq C_n$, $G \neq W_n$, $v_1u_1u_2 \ldots u_s$ is an internal path, and $vw_1 \ldots w_m$ is not an internal path. Let

$$G^* = G - \{v_1u_1, u_1u_2\} + \{v_1u_2, w_mu_1\}.$$

Then $G^* \in \mathscr{T}_n^{k,7}$. By Lemma 2.3, we have $\rho(G^*) > \rho(G)$, a contradiction. Hence $s = 1$, then l_1 is 2. Similarly, we can verify that $l_2 = l_3 = \cdots = l_6 = 2$.

Therefore, we obtain that $G = B_7(1)$. This completes the proof. □

From Lemmas 2.12–2.14 and Theorems 3.1–3.4, we get our main result in this paper.

Theorem 3.5. *Let G be a graph in $\mathscr{T}_n^k$, $k \geqslant 1$. Then $\rho(G) \leqslant \rho(B_3(1))$, and the equality holds if and only if $G \cong B_3(1)$.*

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